Sharp and Simple Bounds for the raw Moments of the Binomial and Poisson Distributions

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Abstract

We prove the inequality $\mathrm{E}[(X/\mu)^k] \le (\frac{k/\mu}{\log(k/\mu+1)})^k \le \exp(k^2/(2\mu))$ for sub-Poissonian random variables, such as Binomially or Poisson distributed random variables with mean μ. The asymptotics $1 + O(k^2/\mu)$ can be shown to be tight for small k. This improves over previous uniform bounds for the raw moments of those distributions by a factor exponential in k.

1. Introduction

Suppose we are given an urn of n balls, each colored red with probability p and otherwise blue. What is the probability that a sample of k balls, with replacement, from this urn has only red balls? Such questions are important in sample efficient statistics and the derandomization of algorithms.

If $X \sim \mathrm{Binomial}(n, p)$ denotes the number of red balls in the urn, the answer is of course given by $P = \mathrm{E}(X/n)^k$. A simple application of Jensen's inequality shows $P \ge p^k$. In the range $k \le \sqrt{np}$, this is nearly tight, as previous bounds (see (1) below) were able to show $P \le C^k p^k$, for some universal constant $C > 1$. In this note we improve the upper bounds on Binomial moments, getting rid of the k in the exponent of C. Specifically we get $P \le p^k e^{k^2/(2pn)}$, which is asymptotically tight with the lower bound as $k = o(\sqrt{np})$.

1.1. Related work

A direct approach to computing the Binomial moments expands them using the Stirling numbers of the second kind [10]: $\mathrm{E}\, X^k = \sum_{i=0}^{k} \left\{ {k \atop i} \right\} n^{\underline{i}} p^i$. This can be derived [6, 11], for example, by combining the much easier to compute factorial moments[1] $\mathrm{E}\, X^{\underline{k}} = n^{\underline{k}} p^k$. Taking the first two terms,

[1] Defined $n^{\underline{k}} = n(n-1)\cdots(n-k+1)$.

one finds that $\mathrm{E}\, X^k = (np)^k \left(1 + \binom{k}{2}\frac{1-p}{np} + O(1/n^2)\right)$ as $n \to \infty$. (See section 2.1 for details.) However, it is not clear what happens when k can not be considered constant with respect to n. Similarly, for the Poisson distribution the moments can be written as the so-called Bell (or Touchard) polynomials [13] in μ: $\mathrm{E}\, X^k = \sum_{i=0}^{k} \left\{ {k \atop i} \right\} \mu^i$. this gives a simple lower bound $\mathrm{E}\, X^k \ge \mu^k(1 + \binom{k}{2}\frac{1}{\mu})$, matching our upper bound for $k = O(\sqrt{\mu})$. However the series does again not easily yield a uniform bound.

Specializing a powerful result on moments of independent random variables by Latała [7], we get for Binomial and Poisson random variables (also Pinelis [9]):

$$\left(c\,\frac{k/\mu}{\log(1+k/\mu)}\right)^k \le \mathrm{E}(X/\mu)^k \le \left(C\,\frac{k/\mu}{\log(1+k/\mu)}\right)^k \tag{1}$$

for some universal constants $c < 1 < C$. Latała's methods are general and allow for bounds of the central moments of similar strength. However, as $k/\mu \to 0$, the upper bound (1) is dominated by the factor C^k, which we manage to remove in this note.

Another option is the use a Rosenthal bound, such as the following by Berend and Tassa [1] (see also Johnson et al. [5])

$$\mathrm{E}\, X^k \le B_k \max\{\mu, \mu^k\}. \tag{2}$$

Here B_k is the kth Bell number, which Berend and Tassa [1] show satisfies $B_k < \left(\frac{0.792k}{\log(k+1)}\right)^k$ and it is known [2, 4] that $B_k^{1/k} = \frac{k}{e\log k}(1 + o(1))$ as $k \to \infty$. The bound is unfortunately incomparable to that of this note, and of Latała, as it behaves as μ rather than μ^k when $\mu < 1$. The connection to Bell numbers is natural since we have exactly $\mathrm{E}\, X^k = B_k = \sum_i \left\{ {k \atop i} \right\}$ for Poisson random variables of $\mu = 1$.

Focusing also on Bell numbers, but using techniques similar to this note, Ostrovsky [8] gave an upper bound for Poisson moments in the case $k \ge 2\mu$:

$$(\mathrm{E}(X/\mu)^k)^{1/k} \le \frac{k/\mu}{e\log(k/\mu)}\left(1 + C(\mu)\frac{\log\log(k/\mu)}{\log(k/\mu)}\right) \quad \text{if } k \ge 2\mu, \tag{3}$$

where $C(\mu) > 0$ is some "constant" depending only on μ. In the range $k < 2\mu$, Ostrovsky only gives the bound $\mathrm{E}(X/\mu)^k \le 8.9758^k$, so similar to the bound of Latała it loses an exponential factor in k compared to Theorem 1 below.

2. Bounds

Theorem 1. *Let X be a non-negative random variable with mean $\mu > 0$ and moment-generating function* $\mathrm{E}[\exp(tX)]$ *bounded by* $\exp(\mu(e^t - 1))$ *for all $t > 0$. Then for all $k > 0$:*

$$\mathrm{E}[X^k] \leq \left(\frac{k}{\log(1 + k/\mu)}\right)^k.$$

A standard logarithmic bound, $\frac{x}{\log(1+x)} \leq 1 + x/2$ (see e.g. [12] eq. 6), implies the corollary

$$\mathrm{E}\left[(X/\mu)^k\right] \leq \left(1 + k/(2\mu)\right)^k \leq \exp\left(k^2/(2\mu)\right).$$

Random variables satisfying the requirement $\mathrm{E}[\exp(tX)] \leq \exp(\mu(e^t - 1))$ are known as sub-Poissonian and include many simple distributions, such as the Poisson or Binomial distribution. We give more examples in Section 3.

Technically our bound is shown using the moment-generating function. This in turn involves some particularly sharp bounds on the Lambert-W function, which is defined by $W(x)e^{W(x)} = x$.[2] We will use the following lemma:

Lemma 1 (Hoorfar and Hassani [3])**.** *For all $y > 1/e$ and $x > -1/e$,*

$$e^{W(x)} \leq \frac{x + y}{1 + \log y}. \tag{4}$$

Proof. Starting from $1 + t \leq e^t$ substitute $\log(y) - t$ for t to get $1 + \log y - t \leq ye^{-t}$. Multiplying by e^t we get $e^t(1 + \log y) \leq te^t + y$. Let $t = W(x)$ s.t. $te^t = x$. Rearranging we get (4). □

Taking $y = e^{W(x)}$ in (4) makes the two sides equal. Hoorfar and Hassani make various substitutions, resulting in different bounds. We will be particularly interested in $y = x + 1$.

Proof of Theorem 1. Let $m(t) = \mathrm{E}[\exp(tX)]$ be the moment-generating function. We will bound the moments of X by

$$\mathrm{E}[X^k] \leq m(t)(k/(et))^k, \tag{5}$$

[2] The Lambert-W function has multiple branches. We are interested in the main one, corresponding to $x > 0$.

which holds for all $k \geq 0$ and $t > 0$. This follows from the basic inequality $1 + x \leq e^x$, where we substitute $tx/k - 1$ for x to get $tx/k \leq e^{tx/k-1} \implies x^k \leq e^{tx}(k/(et))^k$. Taking expectations we get (5).

We now define $B = k/\mu$ and take t such that $te^t = B$. Using the Lambert-W function, this means $t = W(B)$. We note that $t > 0$ whenever $B > 0$.

We can thus bound

$$\begin{aligned}
\mathrm{E}[(X/\mu)^k] &\leq m(t)\mu^{-k}(k/(et))^k \\
&\leq \exp(\mu(e^t - 1)) \left(\frac{k}{e\mu t}\right)^k \\
&= \exp(\mu(B/t - 1)) \left(\frac{e^t}{e}\right)^k \\
&= \exp((k/B)(B/t - 1) + k(t - 1)) \\
&= \exp(kf(B)), \qquad (6)
\end{aligned}$$

where $f(B) = 1/t + t - 1 - 1/B$.

It remains to show $\exp(f(B)) \leq \frac{B}{\log(B+1)}$. Taking logarithms, this means showing the bound

$$1/W(x) + W(x) - 1 - 1/x \leq \log(x) - \log\log(1 + x) \qquad (7)$$

for all $x > 0$, where $W(x)$ is the Lambert-W function.

We define $g(x) = \mathrm{LSH} - \mathrm{RHS} = 1/W(x) + W(x) - 1 - 1/x - \log(x) + \log\log(1 + x)$. For $x \to 0$ we have $1/W(x) = 1/x + 1 + O(x)$ and $\log\log(1 + x) = \log(x) + O(x)$. Thus $g(x) \to 0$ as $x \to 0$. We proceed to show $g(x)$ is non-increasing for $x > 0$, from which (7) follows.

The Lambert function is simple to differentiate using the identity $W'(x) = \frac{W(X)}{x(1+W(x))}$. Taking derivatives and simplifying we get $g'(x) = \frac{1-e^{W(x)}}{x^2} + \frac{1}{(1+x)\log(1+x)}$. Rearranging, it suffices to show the following simple bound on the Lambert function:

$$e^{W(x)} \leq \frac{x^2}{(1 + x)\log(1 + x)} + 1.$$

Using Lemma 1 with $y = x + 1$ yields the bound

$$e^{W(x)} \leq \frac{2x + 1}{1 + \log(1 + x)}.$$

We then need to show $\frac{2x+1}{1+z} \leq \frac{x^2}{(1+x)z} + 1$, where $z = \log(1 + x)$. Solving the quadratic in z, we see that the inequality holds when $x > 0$ and $\frac{x}{1+x} \leq$

$z \leq x$, but we have exactly that inequality for $\log(1+x)$, which finishes the proof. $\square$

2.1. Lower bound

As mentioned in the introduction, the expansion for the Poisson moments $\mathrm{E}\, X^k = \sum_{i=0}^k \left\{ {k \atop i} \right\} \mu^i$ gives a simple lower bound

$$\mathrm{E}\, X^k \geq \mu^k \left(1 + \frac{k(k-1)}{2\mu}\right),$$

matching theorem 1 asymptotically for $k = O(\sqrt{\mu})$. The expansion for Binomial moments $\mathrm{E}\, X^k = \sum_{i=0}^k \left\{ {k \atop i} \right\} n^{\underline{i}} p^i$ yields a similar lower bound

$$\begin{aligned}
\mathrm{E}\, X^k &\geq n^{\underline{k}} p^k + \binom{k}{2} n^{\underline{k-1}} p^{k-1} \\
&= (np)^k \left(\frac{n^{\underline{k}}}{n^k}\right) \left(1 + \binom{k}{2} \frac{1}{(n-k+1)p}\right) \\
&= (np)^k \left(\prod_{i=0}^{k-1} 1 - \frac{i}{n}\right) \left(1 + \binom{k}{2} \frac{1}{(n-k+1)p}\right) \\
&\geq (np)^k \left(1 - \binom{k}{2} \frac{1}{n}\right) \left(1 + \binom{k}{2} \frac{1}{np}\right) \\
&= (np)^k \left(1 + \binom{k}{2} \frac{1-p}{np} \left(1 - \binom{k}{2} \frac{1}{n}\right)\right),
\end{aligned}$$

which matches theorem 1 for $k = O(\sqrt{\mu})$ and p not too close to 1.

We will investigate some more precise lower bounds as k/μ gets large. As briefly mentioned briefly in the introduction, there is a correspondence between the moments of a Poisson random variable and the Bell polynomials defined by $B(k, \mu) = \sum_i \left\{ {k \atop i} \right\} \mu^i$. In particular $\mathrm{E}\, X^k = B(k, \mu)$, if μ is the mean of the Poissonian random variable. The Bell polynomials are so named because $B(k, 1)$ is the kth Bell number. By Dobiński's formula $B(k, 1) = \frac{1}{e} \sum_{i=0}^{\infty} \frac{i^k}{i!}$ the Bell numbers are generalized the real k. We write these as $B_x = B(x, 1)$.

We give a lower bound for $\mathrm{E}(X/\mu)^k$ by showing the following simple connection between the Bell polynomials and Bell numbers:

Theorem 2. *Let k be a positive real number and $\mu \geq 1$ be an integer. Then*

$$B(k, \mu)/\mu^k \geq B_{k/\mu}^{\mu}.$$

While proof below assumes μ is an integer, we will conjecture Theorem 2 to be true for any $\mu \geq 1$. Now by de Bruijn's asymptotic expression for the Bell numbers [2]:

$$E(X/\mu)^k \geq B_{k/\mu}^{\mu} = \left(\frac{k/\mu}{e\log(k/\mu)}(1+o(1))\right)^k \quad \text{as } k/\mu \to \infty.$$

matching the upper bound of Ostrovsky, eq. (3), for large k, as well as Latała's uniform lower bound with a different constant.

Proof of Theorem 2. Let $X, X_1, \dots, X_\mu$ be iid. Poisson variables with mean 1, then $S = \sum_{i=1}^{\mu} X_i$ is Poisson with mean μ. We write $\|X\|_k = (\mathrm{E}\, X^k)^{1/k}$. Then by the AG inequality:

$$\|S/\mu\|_k = \left\|\frac{1}{\mu}\sum_{i=1}^{\mu} X_i\right\|_k \geq \left\|\left(\prod_{i=1}^{\mu} X_i\right)^{1/\mu}\right\|_k = \left(\prod_{i=1}^{\mu} \|X_i\|_k\right)^{1/\mu} = \|X\|_{k/\mu}. \tag{8}$$

Since X has mean 1 we have $\|X\|_{k/\mu} = B_{k/\mu}^{\mu/k}$, and as S has mean μ we have $\|S/\mu\|_k = B(k,\mu)^{1/k}/\mu$. Thus eq. (8) is exactly Theorem 2. $\square$

For small k/μ this bound is less interesting since $B_x \to 0$ as $x \to 0$, rather than 1 as our upper bound. However, it is pretty tight, as we conjecture by the following matching upper bound in terms of the Bell numbers:

Conjecture 1. *For all $k > 0$ and $\mu \geq 1$,*

$$B_{k/\mu}^{1/(k/\mu)} \leq \frac{B(k,\mu)^{1/k}}{\mu} \leq B_{k/\mu+1}^{1/(k/\mu+1)}.$$

Furthermore, for $0 < \mu \leq 1$, $\frac{B(k,\mu)^{1/k}}{\mu} \leq B_{k/\mu}^{1/(k/\mu)}$.

While the upper bound appears true numerically, it can't follow from our moment-generating function bound eq. (6), since it drops below that for k/μ bigger than 40. The conjectured upper bound is even incomparable with our Theorem 1, since it is slightly above $\frac{k/\mu}{\log(1+k/\mu)}$ for very small k/μ. In the region $k < 2$ and $\mu < 1$, the conjectued bound is weaker than eq. (2) [1], but for all other parameters it is substantially tighter.

3. Sub-Poissonian Random Variables

We call a non-negative random variable X sub-Poissonian if $\mathrm{E}\, X = \mu$ and the moment-generating function, mgf., $\mathrm{E}\exp(tX) \le \exp(\mu(e^t - 1))$ for all $t > 0$. If $X_1, \dots, X_n$ are sub-Poissonian with mgf. $m_1(t), \dots, m_n(t)$, then $\sum_i X_i$ is sub-Poissonian as well, since

$$\mathrm{E}\exp\big(t\sum_i X_i\big) = \prod_i m_i(t) \le \prod_i \exp\big(\mu_i(e^t - 1)\big) = \exp\big((\sum_i \mu_i)(e^t - 1)\big).$$

A random variable bounded in $[0, 1]$ with mean μ has mgf.

$$\mathrm{E}\exp(tX) = 1+\sum_{k=1}^{\infty} \frac{t^k E X^k}{k!} \le 1+\mu\sum_{k=1}^{\infty} \frac{t^k E[1^{k-1}]}{k!} = 1+\mu(e^t-1) \le \exp(\mu(e^t-1)).$$

Hence if $X = X_1 + \cdots + X_n$ where each $X_i \in [0, 1]$ we have $\mu = \mathrm{E}\, X = \sum_i \mathrm{E}\, X_i$ and by Theorem 1 that $\mathrm{E}(X/\mu)^k \le \frac{k/\mu}{\log(k/\mu+1)}$. In particular this captures sum of Bernoulli variables with distinct probabilities.

An example of a non-sub-Poissonian distribution is the exponential distribution with mean $1/p = \mu$. The moment generation is $m(t) = \frac{1}{1-\mu(e^t-1)}$, which is larger than $\exp(\mu(e^t - 1))$ for all $t > 0$. It is possible however that similar methods to those in the proof of Theorem 1 will still be applicable.

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